

Recall: $F_A = dA + A \wedge A = \frac{1}{2} F_{ij} dx^i \wedge dx^j \in \Omega^2(\mathfrak{g})$

is the curvature form of A , where

$$F_{ij} = [\nabla_i, \nabla_j] = \partial_i A_j^\alpha - \partial_j A_i^\alpha + A_i^\alpha A_j^\beta - A_j^\alpha A_i^\beta$$

[non-van. curvature $\Rightarrow$ non-triv. connection!]

Conversely:

Theorem: Assume $F_A \equiv 0$. Then there exist local frames in which $A = 0$.

Pf: For $x \in M$, choose a gauge by parallel transport along geodesic rays. $\Rightarrow \nabla_r e_\alpha = 0$

$\Rightarrow$ in polar coordinates, $A_r \equiv 0$

$$0 = F_{r\alpha} = \partial_r A_\alpha - \partial_\alpha A_r + [A_r, A_\alpha]$$

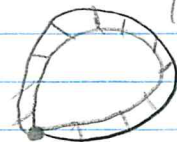
$$0 = \partial_r A_\alpha$$

But $A_\alpha \rightarrow 0$ as $r \rightarrow 0$, so $A_\alpha \equiv 0$. $\square$

Note: in these gauges, the transition func. are constant! $0 = -du u^{-1} + 0$.

$\{ \text{flat connections} \} \xrightarrow{\sim} \{ \text{representations of } \pi_1(M) \}$

$$[x] \mapsto T_x$$



Today: 1. Yang-Mills eqns 2. Chern-Weil Theory 3. Examples 4. Hol. bundle

Additional reference: Jonathan Evans (@UCL) ^{semester} course notes to you.

① small curv. ② more flat than ①
 (curv. = 0) is neutral.

Q. What is the best connection A on a vector bundle E ?

A. One with the least amount of curvature.

Introduce a metric g_{ij} on T^*M . [What does this mean?]

$\Rightarrow g^{ij}$ is a metric on T^*M .

Let φ^i be an orthogonal frame for T^*M .

Define the metric $\langle \cdot, \cdot \rangle$ on $\Omega^k(M)$

so that $|\varphi^i, \dots, \varphi^i| = 1$

(On Euclidean space, $\varphi^i = dx^i$)

Recall: $|F_A|^2 = \frac{1}{2} F_{ij} dx^i dx^j = |F_A|^2$

is the curvature form of A , where

$$F_{ij} = [\nabla_i, \nabla_j] = 2(A_j^x \nabla_x A_i^y - A_i^x \nabla_x A_j^y) + A_i^x A_j^y - A_j^x A_i^y$$

Define the Yang-Mills functional

$$YM(A) = \frac{1}{2} \int_M |F_A|^2 dV \quad \text{[total curvature]}$$

a Yang-Mills connection (best connection)

is a minimizer for YM .

(or, more generally, a stationary point.)

[We need to find out how to do this mathematically.]

$$A^B \otimes I - I \otimes A^B = A^B + -A^B = 0.$$

$$n^x p^x = n^x g^{xx} n^x$$

Can any of these be trivialized?

$$D_A^\dagger F = \star (\omega_A \wedge F)$$

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Variation of y_m :

Change the connection: $A \mapsto A + a$,
 $a \in \Omega^1(\mathfrak{g})$

$$\begin{aligned} \Rightarrow F_A &\mapsto F_{A+a} \\ &= d(A+a) + (A+a) \wedge (A+a) \\ &= dA + da + A \wedge A + a \wedge A + A \wedge a + a \wedge a \\ &= \underbrace{dA + A \wedge A}_{F_A} + \underbrace{da + a \wedge A + A \wedge a}_{t D_A a} + \underbrace{a \wedge a}_{t^2 a \wedge a} \\ \boxed{F_{A+ta} &= F_A + t D_A a + t^2 a \wedge a.} \end{aligned}$$

$$\left. \frac{d}{dt} y_m(A+ta) \right|_{t=0} = \frac{d}{dt} \frac{1}{2} \int \langle F_{A+ta}, F_{A+ta} \rangle dV$$

$$= \frac{d}{dt} \frac{1}{2} \int \left(|F_A|^2 + 2 \langle F_A, t D_A a \rangle + O(t^2) \right) dV$$

$$= \int \langle F_A, D_A a \rangle dV.$$

formal
adjoint

$$= \int \langle D_A^\dagger F_A, a \rangle dV.$$

Conclude: a connection A is Yang-Mills

$$\Leftrightarrow D_A^\dagger F_A = 0 \quad (\text{Yang-Mills equations.})$$

Definition of formal adjoint D_A^* :

Recall the Hodge star operator

$$*: \Omega^k \rightarrow \Omega^{n-k}$$

$$e^{i_1} \wedge \dots \wedge e^{i_k} \mapsto \pm e^{i_{k+1}} \wedge \dots \wedge e^{i_n}$$

= the unique C^∞ -linear map s.t. $\int_M \langle \omega \wedge * \eta, \zeta \rangle dV = \int_M \langle \omega, \zeta \rangle dV$ $\forall \omega, \eta, \zeta \in \Omega^k$

$$\langle \omega \wedge * \eta, \zeta \rangle = \langle \omega, \eta \rangle dV \quad \forall \omega, \eta \in \Omega^k$$

Define $*: \Omega^k(E \otimes E^*) \rightarrow \Omega^{n-k}(E \otimes E^*)$

$$*(\omega \otimes \eta) = (*\omega) \otimes \eta.$$

$n \in \mathbb{Z} \in \mathbb{C}$ { show symmetry

$$\Rightarrow T_n m = m^* \beta \omega^* \beta$$

$$= -m^* \beta \omega^* \beta = -\langle m, \omega \rangle.$$

Therefore

$$-T_n(m \wedge * \omega) = \langle m, \omega \rangle dV$$

for $m, \omega \in \Omega^k(\mathcal{Y})$.

Define $D_A^* = (-1)^{n+k} * D_A *$: $\Omega^{k+1}(\mathcal{Y}) \rightarrow \Omega^k(\mathcal{Y})$

We claim that this satisfies the desired

property $\int_M \langle m, D_A \omega \rangle dV = \int_M \langle D_A^* m, \omega \rangle dV$

for smooth forms $m \in \Omega^{k+1}(\mathcal{Y})$, $\omega \in \Omega^k(\mathcal{Y})$.

Proof: $\int \langle m, D_A w \rangle dV = - \int T_n(D_A w \wedge x_n) dV$

Note: $d(T_n w \wedge x_n) = T_n(D_A w \wedge x_n) + (-1)^k T_n(w \wedge D_A x_n)$

NAD for GOTTW

(-1) sign & integrate both sides:

IBP.

above $= (-1)^k (-1)^k \int T_n(w \wedge D_A x_n)$

Exercise:

$\star^2 = (-1)^{k(m-k)}$ on Ω^k .

$\psi^1 \wedge \dots \wedge \psi^k \mapsto \psi^{p+1} \wedge \dots \wedge \psi^m$
 $\mapsto \star \psi^1 \wedge \dots \wedge \psi^k$
 $\star \psi^{p+1} \wedge \dots \wedge \psi^m \mapsto \psi^1 \wedge \dots \wedge \psi^k$
 $= \psi^1 \wedge \dots \wedge \psi^m$

$= -(-1)^k (-1)^{k(m-k)} \int \langle w, \star D_A x_n \rangle$

$(-1)^{k(m-k+1)+k} = (-1)^{k \cdot m + 1}$

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南

note: easier formula: $\int \langle u, \nabla \cdot w \rangle dV = - \int \langle \nabla \cdot u, w \rangle dV$

Example: Y^m on $\mathbb{R}^n / \mathbb{R}^n$ (too) $\nabla^* u^i = - \nabla_i u^i$
 Do examples. $\star F = \text{vol}$ $\nabla^* F = - \nabla_i F_i$

~~all the stuff about the connection and the curvature is not needed here~~

[best in dim 4.]

[Want to get some lower bounds on YM energy.]

[Left: Chern-Weil, degree = $\int c_2$, hol c , degree of a point bundle. (compute from any sec. section?)]

Chern-Weil Theory:

assume E is a $U(n)$ -bundle.

$U(n) = \frac{O(2n)}{GL(n, \mathbb{C})} \subset GL(2n, \mathbb{R})$

$\Rightarrow$ well-defined $\mathbb{C}$ -action on fibers

$\Rightarrow$ local frames $\{e_i\}_{i=1}^n$ spanning / E

• Hermitian i.p. $\langle \cdot, \cdot \rangle$ w/ $\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta}$

$$\langle \lambda s, t \rangle = \lambda \langle s, t \rangle$$

$$\langle s, t \rangle = \overline{\langle t, s \rangle}$$

$$T_n = T_n^c$$

Lemma (Chern-Weil). Let A be any unitary connection on E . Then the cohomology class $[T_n F_A \wedge \dots \wedge F_A] \in H_{DR}^{2n}(M, \mathbb{C})$ is independent of A .

Proof: let $B = A + a$ be another conn.

$$k_A = \sum t^p \wedge \dots \wedge t^p$$

$$22 \log |2|/h$$

Consider the path $A_t = A + ta \in \mathcal{A}_E$.

$$\text{Recall } \frac{d}{dt} F_{A_t} = \frac{d}{ds} \left(F_{A_t} + D_{A_t} s a + s^2 a \wedge a \right) \Big|_{s=0}$$

①

$$= D_{A_t} a$$

$$\text{Now: } \frac{d}{dt} T_n[F_{A_t} \wedge \dots \wedge F_{A_t}] \stackrel{\text{cyclic}}{=} n T_n[D_{A_t} a \wedge F_{A_t} \wedge \dots \wedge F_{A_t}]$$

$$\text{Bianchi: } d T_n[a \wedge F_{A_t} \wedge \dots \wedge F_{A_t}]$$

$$= T_n[D_{A_t} a \wedge F_{A_t} \wedge \dots \wedge F_{A_t}] + T_n[a \wedge D_{A_t} F_{A_t} \wedge \dots \wedge F_{A_t}]$$

Coroll: $\Rightarrow \frac{d}{dt} \text{Tr}[F_{A_1} \wedge \dots \wedge F_{A_n}] = p \cdot d \text{Tr}[a_1 F_{A_1} \wedge \dots \wedge F_{A_n}]$

cor: $\text{Tr}[F_B \wedge \dots \wedge F_B] - \text{Tr}[F_A \wedge \dots \wedge F_A]$
 $= d \left[p \cdot \int_0^1 \text{Tr}[a_1 F_{A_1} \wedge \dots \wedge F_{A_n}] dt \right]$

□

Define the Chem Classes -

$$c_i(E) \in H^{2i}(M, \mathbb{Z}), \quad i=1, \dots, n = \text{rk } E$$

by the prescription

$$\left[\text{Tr} \Lambda^n \left(I + \frac{it}{2\pi} F_A \right) \right] = \sum_{i=1}^n c_i(E) \frac{t^i}{i!}$$

in particular:

$$c_1(E) = \left[\frac{i}{2\pi} \text{Tr} F_A \right] \text{ is the first Chem class}$$

Corollary: (i) $c_1(E \oplus F) = c_1(E) + c_1(F) = c_1(E \oplus F)$

(ii) $c_1(E \otimes F) = \text{rk}(E) \cdot c_1(F) + \text{rk}(F) \cdot c_1(E)$

Pf of (ii): $A \otimes B \cong A \otimes I_F + I_E \otimes B$ is a

connection on $E \otimes F \Rightarrow F_{A \otimes B} = F_A \otimes I_F + I_E \otimes F_B$ □

Ex: Let M be a compact oriented surface.

Define the degree: $\deg(E) = \int_M c_1(E)$
 $= \frac{1}{2\pi i} \int_M \text{Tr} F_A$

Fact: This number determines E , as a $\text{GL}(n, \mathbb{C})$ -bundle (Chern theory.)

Examples.

$$\text{Lie}(\mathfrak{u}(1)) = i \cdot \mathbb{R}$$

1. Let $M = \mathbb{R}^3$, $A = i(A_1 dx^1 + A_2 dx^2 + A_3 dx^3)$

a $U(1)$ -connection $\vec{A} = (A_1, A_2, A_3)$

$$F_A = dA = i(\partial_1 A_2 - \partial_2 A_1) dx^1 \wedge dx^2 +$$

$$+ i(\partial_1 A_3 - \partial_3 A_1) dx^1 \wedge dx^3 +$$

$$+ i(\partial_2 A_3 - \partial_3 A_2) dx^2 \wedge dx^3$$

$$\vec{\nabla} \times \vec{A} = \vec{B} \Leftrightarrow$$

$$\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ A_1 & A_2 & A_3 \\ i & j & k \end{pmatrix}$$

$$(F_{ij}) = i \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}$$

$$*F = i(B_1 dx^1 + B_2 dx^2 + B_3 dx^3)$$

Yang-Mills: $D^*F = 0 \Leftrightarrow \vec{\nabla} \times \vec{B} = 0$

Bi-anchi: $DF = 0 \Leftrightarrow \vec{\nabla} \cdot \vec{B} = 0$.

$\Rightarrow$ Maxwell's eqns w/ $E \equiv 0$!

Exercise: you on $\mathbb{R}^{3,1}$ given full Maxwell eqns.

Note: Yang-Mills on $SU(2)$ -bundles over

Riemannian 4-mfds = Donaldson Theory!

Exercise in Chen-Weil theory: $\text{ym}(A) \geq 4\pi^2 c_2(E)$

in this case!

you on d.s.: Before the degree.

Note: $D^*F = 0 \Rightarrow D_A *F = 0 \Rightarrow \vec{\nabla}_A *F = 0$

consider $*F = \lambda I \dots$

Too trivial. ≤ 0

Holomorphic vector bundles Reference: Griffiths & Harris

A holomorphic vector bundle, E , is a holomorphic submersion $\pi: E \rightarrow M$ with the following properties:

- (i) Each fiber $\pi^{-1}(x)$ has the structure of a $\mathbb{C}$ -vector space [compatible w/ cpx str. on M].
- (ii) Each point x has a nbhd $U \ni x$ and a local hol'c frame $\{f_\mu\}$ of hol'c sections.

$\rightarrow$ any holomorphic section $s: U \rightarrow E$

$$\text{can be written } s = \sum_{\mu=1}^n s^\mu f_\mu$$

for unique holomorphic functions s^μ .

Notes:
 The transition func g^μ_ν are holomorphic functions matrix-valued.
 We write E for the underlying C^∞ (complex) vector bundle.
 Holomorphic bundles are more difficult!

Reason: For any C^∞ bundle there is local sections $\rightarrow$ global sections

$$\text{a map } T(U, E) \rightarrow T(M, E)$$

$\varphi \in C^\infty$, $\text{supp } \varphi \subset U$.

$$s \mapsto \varphi \cdot s$$

for any open set $U \subset M$.

[local sections extend to global.]

But s hol'c $\nRightarrow \varphi \cdot s$ hol'c!

$T(M, E)$ is very difficult to compute. [Ext. chom. !]

Fortunately, the extra structure

has an advantage:

Recall: on any complex mfd,

differential forms split into type (p,q) :

$$\Lambda^{p+q} T^*M \otimes \mathbb{C} \subset \Omega_c^{p,q} = \langle dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q} \rangle.$$

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \quad dz = \frac{1}{2}(dx + idy)$$

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \quad d\bar{z} = \frac{1}{2}(dx - idy)$$

On $\Omega^{p,q}$, we have $\bar{\partial} = \pi_{p,q+1} \circ d$

$$\text{Define } \bar{\partial}_E: \Omega^{p,q}(E) \rightarrow \Omega^{p,q+1}(E)$$

by

$$\bar{\partial}_E(\omega \otimes f_\mu) := \bar{\partial}\omega \otimes f_\mu.$$

This is well-defined!

$$\bar{\partial}_E(\omega \otimes f_\nu) = \bar{\partial}_E(\omega \otimes g^\mu_\nu f_\mu)$$

$$= \bar{\partial}_E((g^\mu_\nu \omega) \otimes f_\mu) = (\bar{\partial}(g^\mu_\nu \omega) \otimes f_\mu$$

$$+ g^\mu_\nu \bar{\partial}\omega \otimes f_\mu)$$

$$= \bar{\partial}\omega \otimes g^\mu_\nu f_\mu$$

$$= \bar{\partial}\omega \otimes f_\nu \quad \checkmark$$

[In C^∞ case, to get D_A , needed a connection!]

$$\text{Note: } \bar{\partial}_E^2(\omega \otimes f_\mu) = \bar{\partial}_E^2 \omega \otimes f_\mu = 0.$$

$$\Rightarrow \bar{\partial}_E^2 = 0.$$

$$\bar{\partial}^2 = 0 \Rightarrow \bar{\partial}^2 = 0$$

$$\begin{array}{c} \Omega^{p,q} \\ \swarrow \quad \searrow \\ \Omega^{p+1,q} \quad \Omega^{p,q+1} \\ \swarrow \quad \searrow \\ \Omega^{p+2,q} \quad \Omega^{p,q+2} \end{array}$$

Why does a Hermitian metric give also a vol?

trivially, because...

Now, let $\langle \cdot, \cdot \rangle$ be a hermitian metric on E . $\langle \lambda s, t \rangle = \lambda \langle s, t \rangle$ $\langle s, \lambda t \rangle = \bar{\lambda} \langle s, t \rangle \Rightarrow \det C_{i,j} = 1$ (assuming metric)

~~Complex geom. not. then:~~ $\langle s, t \rangle = s^\mu h_{\mu\bar{\nu}} t^{\bar{\nu}}$ $\mu, \nu \in \{1, \dots, n\}$

~~$h_{\mu\bar{\nu}} = h_{\bar{\nu}\mu}$~~

~~$\mu, \nu \in \{1, \dots, n\}$~~

~~Define $h^\mu_{\bar{\nu}} = (h_{\bar{\nu}\mu})$ s.t.~~

~~$h_{\mu\bar{\nu}} h^\mu_{\bar{\rho}} = \delta_{\bar{\nu}\bar{\rho}}$~~

~~Taken to heart: E on $g^\mu_\nu = g^\nu_\mu \Rightarrow$~~

Complex geometry notation: h is a section of $E^* \otimes \bar{E}^*$

$E^* \otimes \bar{E}^*$

g^μ_ν $g^{\mu\nu}$

Exercise: Complex geometry notation: use hol. frame

Chern connection is given

by $A^\nu_{j\mu} = 0$

$A^\nu_{j\mu} = \frac{1}{2} (\partial_j h_{\mu\bar{\nu}}) h^{\bar{\nu}\nu}$

Curvature: $\partial_i A^\nu_{j\mu} - \partial_j A^\nu_{i\mu} = F_{ij}^{\nu\mu}$

Definition: We say that a unitary connection ∇_A is compatible with a hermitian metric on a hol. v.b. E if:

(i) [It is a metric connection:]

$\partial_r \langle s, t \rangle = \langle \nabla_r s, t \rangle + \langle s, \nabla_r t \rangle$

$\forall (v \in TM, s, t \in \Gamma(E))$

(ii) $D_A^{0,1} := \pi^{0,1} \circ D_A = \bar{\partial}_E$

real tangent vectors!

Lemma: (Chern) : Given $\bar{\partial}_E$ and $\langle \cdot, \cdot \rangle$ determine a unique compatible unitary connection A , called the Chern connection.

pf: Let $\{e_\alpha\}_{\alpha=1}^n$ be a unitary frame for E . Write

$$\bar{\partial}_E e_\alpha^\perp = a^\beta_{\alpha} e_\beta$$

$\Rightarrow a$ is a local matrix-valued $(0,1)$ -form

if A is unitary then

$$0 = \bar{\partial}_E \langle e_\alpha, e_\beta \rangle = \langle \bar{\partial}_E e_\alpha, e_\beta \rangle$$

real tangent vector! $\langle e_\alpha, \bar{\partial}_E e_\beta \rangle$

$$= \langle \bar{\partial}_E e_\alpha, e_\beta \rangle + \langle \bar{\partial}_E e_\beta, e_\alpha \rangle$$

$$= A^\alpha_{\nu\beta} + A^\beta_{\nu\alpha}$$

$$\Rightarrow 0 = A + A^* \quad (A \text{ is skew-Hermitian})$$

$$= A^{1,0} + a + (A^{1,0})^* + a^{1,*}$$

type $(0,1)$
type $(1,0)$

$$\text{Define } A^{1,0} = -a^* \Rightarrow a = -(A^{1,0})^* \quad \checkmark$$

$$\Rightarrow A = a - a^* \quad \square$$

Note: curvature of a Chern conn. is a $(1,1)$ form.

Converse is also true!

Return to cpx geom exercise!

$$DA = \dots$$